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SPARSE RAMSEY GRAPHS

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If H is a Ramsey graph for a graph G then H is rich in copies of the graph G. Here we prove theorems in the opposite direction. We find examples of H such that copies of G do not form short cycles in H. This provides a strenghtening also, of the following well-known result of Erdős: there exist graphs with high chromatic number and no short cycles. In particular, we solve a problem of J. Spencer.

1. Introduction

Throughout this paper we use the term "subgraph" to mean *induced* subgraph. We write $G \leqq G'$ to indicate that G is a subgraph of G'. If G and H are graphs then we denote by $\binom{H}{G}$ the set of all subgraphs of H which are isomorphic to G.

Let $\mathcal{G}$ be a subset of $\binom{H}{G}$. Such a set $\mathcal{G}$ is called a *system of copies of* G in H.

Definition 1.1. $\mathcal{G} \subseteqq \binom{H}{G}$ *is a t-Ramsey system of copies of G if for every partition* $\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_t$ *of the edges of H into t classes there exist* $G' \in \mathcal{G}$ *and* $i \in \{1, 2, \ldots, t\}$ *such that* $E(G') \subseteqq \mathcal{A}_i$.

Alternatively, a partition $\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_t$ of the edges of H will sometimes be considered as a colouring $\varphi: E(H) \to \{1, 2, \ldots, t\}$ defined by $\varphi(e) = i$ iff $e \in \mathcal{A}_i$.

If $\mathcal{G} = \binom{H}{G}$ is t-Ramsey then we simply write $H \to (G)_t$.

Let $\mathcal{G} \subseteqq \binom{H}{G}$ be a system of copies of G in H. Put $\mathcal{G}_e = \{E(G');\ G' \in \mathcal{G}\}$, i.e., the system of edge sets of copies from $\mathcal{G}$.

Let $\mathcal{S} = (X, \mathcal{M})$ be a set system, i.e., $\mathcal{M} \subseteqq \mathcal{P}(X)$. Recall that the chromatic number $\chi(X, \mathcal{M})$ of $(X, \mathcal{M})$ is the minimal number of classes $X_1, \ldots, X_k$ of a partition of X such that no X_i contains an edge $M \in \mathcal{M}$.

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Sometimes we specify the set system by the edge set $\mathcal{M}$ only. In that case we mean the set system $(\bigcup \mathcal{M}, \mathcal{M})$. Recall that a cycle of length l in a set system $(X, \mathcal{M})$ is an alternating sequence $x_0, M_1, x_1, M_2, \ldots, x_n$ of vertices and edges of $(X, \mathcal{M})$ satisfying $x_n = x_0$, $M_i \supseteq \{x_{i-1}, x_i\}$ with at last two vertices x_i, x_j and two distinct edges M_i and M_j. Sometimes we specify a cycle by means of the edge-sequence only.

The Lemma below follows immediately from comparison of the definitions:

Lemma 1.2. $H \xrightarrow{\mathcal{G}} (G)_t$ *iff* $\chi(\mathcal{G}_e) > t$. ∎

We prove here:

Theorem 1.3. *For every pair of positive integers l, t and for every graph G there exist a graph H and a system $\mathcal{G}$ of copies of G in H such that*

1. $H \xrightarrow{\mathcal{G}} (G)_t$

2. $\mathcal{G}_e$ *does not contain any cycle of length less then l.*

Theorem 1.4. *For every triple of positive integers l, t, n there exists a graph H such that*

1. $H \to (K(n))_t$
2. $\binom{H}{K(n)}_e$ *does not contain any cycle of length less than l.*

Remark that Theorem 1.3 generalizes results from [2, 4, 6] and Theorem 1.4 answers a question of J. Spencer [8]. As the proofs of both theorems are constructive this provides a strenghtening of [8] as well.

The above theorems were announced in [7] and they also strenghten the construction of graphs with high chromatic number and without short cycles [3]. Our proof is based on the so-called *partite construction* introduced in [6] (which, incidentally, yields a short proof of the existence of graphs with high chromatic number and no short cycles, see [5]).

This paper consists of three parts. In Section 1 we give definitions and outline the strategy of our proofs. In Section 2 we prove Theorems 1.3–4. Section 3 contains some concluding remarks.

2. Preliminaries

In this part we introduce the notion of an a-partite graph and related special symbols $\mathcal{B} * ((X_i)_{i=1}^a, E)$, $B * ((X_i)_{i=1}^a, E)$, $\mathcal{B} * \mathcal{G}$. They will be used in the proof of the main theorem.

Definition 2.1. *An a-partite graph is a pair $((X_i)_{i=1}^a, E)$ where the sets X_i form a partition of the vertex set $\bigcup_{i=1}^a X_i$ and $\left(\bigcup_{i=1}^a X_i, E\right)$ is a graph such that no X_i contains any edge from E.*

We remark that we allow $X_i = \emptyset$ for some i.

Two a-partite graphs $((X_i)_{i=1}^a, E)$ and $((X_i')_{i=1}^a, E')$ are said to be *isomorphic* if there exists a bijection $f\colon \bigcup_{i=1}^a X_i \to \bigcup_{i=1}^a X_i'$ which satisfies $f(X_i) = X_i'$ for every $i = 1, \ldots, a$ and $\{f(X), f(y)\} \in E'$ iff $\{x, y\} \in E$.

$((X_i)_{i=1}^a, E)$ is said to be an (induced) *subgraph* of $((X_i')_{i=1}^a, E')$ if $X_i \subseteq X_i'$ for every $i=1, \ldots, a$ and if $\left(\bigcup_{i=1}^{a} X_i, E\right)$ is an (induced) subgraph of $\left(\bigcup_{i=1}^{a} X_i', E'\right)$.

A subgraph of $((Y_i)_{i=1}^a, F)$ isomorphic to $((X_i)_{i=1}^a, E)$ will be often referred to as a *copy* of $((X_i)_{i=1}^a, E)$ in $((Y_i)_{i=1}^a, F)$.

Definition 2.2. *Let $((X_i)_{i=1}^a, E)$ be an a-partite graph. Fix $c, d \in \{1, \ldots, a\}$, $c \neq d$. Let $((Y_c, Y_d), F')$ be a bipartite graph and let $\mathscr{B}$ be a family of copies of $((X_c, X_d), E')$ in $((Y_c, Y_d), F')$, where $E' = \{e \in E;\ e \subseteq X_c \cup X_d\}$. Put $\mathscr{B} = \{B_1, \ldots, B_r\}$. For each copy B_j let $\varphi_j\colon X_c \cup X_d \to Y_c \cup Y_d$ be the corresponding isomorphism (i.e. $\varphi_j(X_c) \subseteq Y_c$, $\varphi_j(X_d) \subseteq Y_d$). We define an a-partite graph $\mathscr{B} * ((X_i)_{i=1}^a, E) = ((Y_i)_{i=1}^a, F)$ as follows: $Y_i = X_i \times \{1, \ldots, r\}$ for $i \neq c, d$; $\{\alpha, \beta\} \in F$ if one of the following possibilities holds:*

(i) $\alpha = (x, j)$, $\beta = (y, j)$ *and* $\{x, y\} \in E$;

(ii) $\beta = (y, j)$, $\alpha = \varphi_j(x)$ *and* $\{x, y\} \in E$ *for some* $x \in X_c \cup X_d$;

(iii) $\alpha \in Y_c$, $\beta \in Y_d$ *and* $\{\alpha, \beta\} \in F'$.

Intuitively, $\mathscr{B} * ((X_i)_{i=1}^a, E)$ is an amalgamation of copies of $((X_i)_{i=1}^a, E)$ along the set system $\mathscr{B}$.

We introduce some further notation and mention a number of trivial facts:

(a) For $j=1, \ldots, r$ denote by $\Psi_j\colon \bigcup_{i=1}^{a} X_i \to \bigcup_{i=1}^{a} Y_i$ the 1—1 mapping defined by

$$\Psi_j(x) = \varphi_j(x) \quad \text{for} \quad x \in X_c \cup X_d$$

$$\Psi_j(x) = (x, j) \quad \text{for} \quad x \notin X_c \cup X_d.$$

(b) It is easily seen that

$$F = \big\{\{\Psi_j(x), \Psi_j(y)\}\colon\ \{x, y\} \in E,\ j \leqq r\big\}$$

and that for each B_j the subgraph of $\mathscr{B} * ((X_i)_{i=1}^a, E)$ induced by the set $\left\{\Psi_j(x)\colon\ x \in \bigcup_{i=1}^{a} X_i\right\}$ is isomorphic to $((X_i)_{i=1}^a, E)$.

(c) *(Notation.)* This a-partite graph will be denoted by $B_j * ((X_i)_{i=1}^a, E)$.

(d) Explicitly $B_j * ((X_i)_{i=1}^a, E) = \big((\Psi_j(X_i))_{i=1}^a,\ \{\{\Psi_j(x), \Psi_j(y)\}\colon \{x, y\} \in E\}\big)$.

(e) *(Notation.)* If $G = ((X_i^v)_{i=1}^a, E^v)$ is an a-partite subgraph of $((X_i)_{i=1}^a, E)$ then $B_j * G$ will stand for the Ψ_j-image of G.

(f) Explicitly, $B_j * G = ((Y_i^v)_{i=1}^a, F^v)$, where $Y_i^v = \Psi_j(X_i^v)$ for $i=1, \ldots, a$ and $F^v = \{\{\Psi_j(x), \Psi_j(y)\}\colon \{x, y\} \in E^v\}$.

(g) *(Notation.)* If $\mathscr{G}$ is a system of subgraphs of $((X_i)_{i=1}^a, E)$ then $\mathscr{B} * \mathscr{G}$ denotes the set of all subgraphs of $\mathscr{B} * ((X_i)_{i=1}^a, E)$ of the form $B_j * G'$, $B_j \in \mathscr{B}$, $G' \in \mathscr{G}$.

(h) The bipartite graph $((X_c, X_d), E')$ may be considered as an a-partite graph $((X_i')_{i=1}^a, E')$ where $X_c' = X_c$, $X_d' = X_d$, $X_i' = \emptyset$ otherwise. Using this convention $\mathscr{B}$ is a family of subgraphs of $\mathscr{B} * ((X_i)_{i=1}^a, E)$. In particular, B_j is a subgraph of $B_j * ((X_i)_{i=1}^a, E)$ for every $j=1, \ldots, r$.

Theorems 1 and 2 will be deduced from the following somewhat technical statement:

Proposition 2.3. *For every pair of positive integers t, l and every bipartite graph G there exist a bipartite graph H and a family* $\mathcal{G} \subseteq \binom{H}{G}$ *with the following properties:*

1. $H \xrightarrow{\mathcal{G}} (G)_t$;
2. $\mathcal{G}_e$ *does not contain cycles of length less than or equal to* l;
3. *If* $l > 2$ *and* G_1, G_2 *are two distinct members of* $\mathcal{G}$ *then* G_1 *and* G_2 *intersect in at most 2 vertices. Moreover, if* x, y *are common vertices of* G_1 *and* G_2 *then* $\{x, y\}$ *is an edge (in both* G_1 *and* G_2*).*

Observe that 3 is slightly stronger than the fact that G_e does not contain any 2-cycle.

We will prove the Proposition by induction on l. Theorem 1.3 will be established by the same construction (and using Proposition 1). Theorem 1.4 is a direct consequence of the proof of Theorem 1.3 given below.

3. Proofs

Proof of Proposition 2.3. We proceed by induction on l. The case $l=1$ (i.e. the induced bipartite Ramsey Graph-Theorem) is easy and folkloristic. For the sake of completeness, note that any bipartite graph is a subgraph of a bipartite graph of type $(X, [X]^p;\ \{\{x, P\} : x \in P \in [X]^p\}) = (X, [X]^p;\ \in)$ for some positive integer p and set X.

Using Ramsey's theorem we get that for every p, X there are p', X' such that

$$H = (X', [X']^{p'}, \in) \to (G)_t \quad \text{where} \quad G = (X, [X]^p, \in).$$

Clearly, $\mathcal{G} = \binom{H}{G}$ satisfies the condition $(l=1)$ (see [6] for more details).

In the induction step assume that Proposition 2.3 is valid for a fixed $l \geqq 1$. Fix $G=(V, E)$, $|V|=n$ and a positive integer t. Without loss of generality assume $n \geqq 3$. Let $H'=(V', E')$ be a bipartite graph with $H' \to (G)_t$ (which exists by the case $l=1$). Suppose further

$$V' = \{1, 2, \ldots, a\}$$

$$E' = \{e_1, \ldots, e_m\}$$

$$\mathcal{G}' = \binom{H'}{G} = \{G_1, \ldots, G_q\}.$$

Define inductively an a-partite graph P^k and a system $\mathcal{G}^k$ for all $k \leqq m$ as follows:

$$P^0 = ((X_i^0)_{i=1}^a, E^0)$$

where

$$X_i^0 = \{i\} \times \{1, \ldots, q\}$$

and

$$\{(v, j), (v', j')\} \in E^0 \quad \text{iff} \quad j = j' \quad \text{and} \quad \{v, v'\} \in E(G_j).$$

Denote by $\bar{G}_j$ the subgraph of P^0 induced by the vertices $V(G_j) \times \{j\}$ and put $\mathcal{G}^0 = \{\bar{G}_j;\ j=1, \ldots, q\}$.

Suppose that $P^k=((X_i^k)_{i=1}^a, E^k)$ and $\mathcal{G}^k \subseteq \binom{P^k}{G}$ have already been defined for $k<m$. Put $\{c,d\}=e_{k+1}\in E'$ and consider the bipartite graph:

$$B^{k+1}=(X_c^k, X_d^k, E'), \quad \text{where} \quad E'=\{e\in E:\ e\subseteq X_c^k\cup X_d^k\}.$$

Now, applying the induction hypothesis to the graph B^{k+1}, we obtain that there exist a bipartite graph $C^{k+1}=(Y_c, Y_d, F')$ and and system $\mathcal{B}^{k+1}\subseteq\binom{C^{k+1}}{B^{k+1}}$ such that

(i) $C^{k+1}\xrightarrow{\mathcal{B}^{k+1}}(B^{k+1})_t$,

(ii) $(\mathcal{B}^{k+1})_e$ does not contain cycles of length $<l$,

(iii) singletons and edges are the only pairwise intersections of copies from $\mathcal{B}^{k+1}$ (cf. 3. in Proposition 2.3).

Put

$$P^{k+1}=\mathcal{B}^{k+1}*P^k \quad \text{and} \quad \mathcal{G}^{k+1}=\mathcal{B}^{k+1}*\mathcal{G}^k.$$

Finally, let

$$H=P^m=((Y_i)_{i=1}^a, F) \quad \text{and} \quad \mathcal{G}=\mathcal{G}^m.$$

We will prove that H and G have all the desired properties.

Claim 1. $H\xrightarrow{\mathcal{G}}(G)_t$.

Proof. Following the lines of [6], [9] we use backwards induction on $k=m, m-1, \dots$ $\dots, 0$. Let $\varphi: E(H)\to\{1,\dots,t\}$ be an arbitrary coloring. Using the definition of P^m there exists a $\bar{B}^m\in\mathcal{B}^m$ such that φ restricted to the edge set of $\bar{B}^m$ is a constant (say φ^m). (Note that $\bar{B}^m\in\mathcal{B}^m$ is a subgraph of C^m isomorphic to B^m. The upper index indicates that $\bar{B}^m$ is related to the a-partite graph P^m and thus also to the edge e_m of the graph H'.) Consider the restriction of φ to the edge set of the graph $\bar{B}^m*p^{m-1}$ (which is isomorphic to P^{m-1}).

Using the definition of P^{m-1} there exists $\bar{B}^{m-1}\in\mathcal{B}^{m-1}$ such that φ restricted to the edge set of $\bar{B}^m*\bar{B}^{m-1}$ is constant (say φ^{m-1}). Proceeding this way we obtain $\bar{B}^i\in\mathcal{B}^i$ $i=m, m-1, \dots, 1$ such that the edge set of $\bar{B}^m*(\bar{B}^{m-1}*\dots*(\bar{B}^{i+1}*\bar{B}^i)\dots)$ is coloured by φ^i (cf. Section 2(e)). Now consider $P'=\bar{B}^m*(\dots*(\bar{B}^1*P^0)\dots)$, P' is a copy of P^0. Observe that for every edge $\{i,i'\}=e_j\in E'$ the following holds: the set of all edges e of P' which satisfy $e\subseteq Y_i\cup Y_{i'}$ is a subset of the edge set of $\bar{B}^m*(\dots*(\bar{B}^{j+1}*\bar{B}^j)\dots)$. Consequently, the colour of an edge e of P' depends only on those sets $Y_i, Y_{i'}$ for which $e\subseteq Y_i\cup Y_{i'}$. This induces a coloring φ' of the edge set E' of H' by $\varphi'(e_j)=\varphi^j$ for $j=1,2,\dots,m$.

Thus, there exists $G'\in\mathcal{G}'$ such that the coloring φ' restricted to the edge set of G' is a constant. Hence, the coloring φ restricted to the edge set of a copy $\bar{B}^m*(\bar{B}^{m-1}*\dots*(\bar{B}_1^1*\bar{G}')\dots)$ is constant. This proves Claim 1. ∎

Claim 2. *H is bipartite.*

This is obvious as H' is bipartite and the bipartition of H' induces the bipartition on each of $P^k, k=0,\dots,m$ (the mapping $f\colon \bigcup_{i=1}^{a} X_i^k\to\{1,\dots,a\}$ defined by $f(x)=i$, for all $x\in X_i^k$, is a homomorphism, $f\colon P^k\to H'$).

Claim 3. *Any two distinct copies from $\mathcal{G}$ intersect in at most two vertices. If they intersect in exactly two vertices then these vertices form an edge (in both copies).*

Proof. Follows easily by induction. The case P^0, $\mathcal{G}^0$ is trivial. Put $e_{k+1}=\{c, d\}$. If G_1, G_2 are members of $\mathcal{G}^{k+1}=\mathcal{B}^{k+1}*\mathcal{G}^k$ then put $G_i=B_i*\bar{G}_i$, $B_i\in\mathcal{B}^{k+1}$, $\bar{G}_i\in\mathcal{G}^k$ for $i=1, 2$. If now $B_1=B_2$ then we use induction on k and if $B_1\neq B_2$ then the vertices of the intersection $G_1\cap G_2$ are contained in the set $Y_c\cup Y_d$. Moreover, as G_1 and G_2 correspond to copies of G in H' and as $\{c, d\}$ is an edge of H' we get that $|V(G_i)\cap(Y_c\cup Y_d)|\geqq 2$ iff $|E(G_i)\cap[Y_c\cup Y_d]^2|=1$ for both $i=1, 2$. This proves Claim 3. ▮

Claim 4. *$\mathcal{G}_e$ does not contain cycles of length $<l+1$.*

Proof. Clearly, we may assume $l>2$ as the case $l=1$ is trivial and the case $l=2$ follows from Claim 3. We proceed by induction proving that $(\mathcal{G}^k)_e$ does not contain cycles of length $<l+1$ for $k=0, 1, \ldots, m$. This is obvious for $(\mathcal{G}^0)_e$ as $\mathcal{G}^0$ is a collection of q disjoint copies of G. Assume that $(\mathcal{G}^k)_e$ does not contain cycles of length $<l+1$ for some $k\geqq 0$.

Suppose for a contradiction that the sequence $E(G_1), \ldots, E(G_{e'})$ of the edge sets of these graphs $G_1, \ldots, G_{e'}$ which belong to $\mathcal{G}^{k+1}$ form a cycle of length $l'<l+1$. Denote by E_i the edge set of G_i. We may assume that all the sets E_i are distinct and that $E_{i-1}\cap E_i\neq E_i\cap E_{i+1}\neq\emptyset$ for $i=1, \ldots, l'$ (cyclically).

Recall that $\mathcal{G}^{k+1}=\mathcal{B}^{k+1}*\mathcal{G}^k$. Thus each G_i, $i=1, \ldots, l'$ is of the form

$$G_i = B_i * \bar{G}_i,$$

where $\bar{G}_i\in\mathcal{G}^k$ and $B_i\in\mathcal{B}^{k+1}$. Observe that both $\bar{G}_i$ and B_i are uniquely determined by G_i (as G has at least 3 vertices) for every $i=1, \ldots, l'$. Moreover, if $G_i=B_i*\bar{G}_i$, $G_j=B_j*\bar{G}_j$ and $B_i\neq B_j$ when $E_i\cap E_j\neq\emptyset$, then B_i and B_j intersect in (exactly) one edge (as $\mathcal{B}^{k+1}$ satisfies condition 3. of Proposition 2.3). Consequently the edge sets of the graphs $B_1, \ldots, B_{l'}$ either coincide or contain a cycle of length $<l+1$.

The first case is impossible, for if $B=B_1=\ldots=B_{l'}$ then $G_i=B*\bar{G}_i$ and thus $\bar{G}_i$ form a cycle of length $<l+1$ in $\mathcal{G}^k$, a contradiction.

Now consider the second case. Denote by $\tilde{E}_i$ the edge set of B_i. Observe that $|E_i\cap\tilde{E}_i|=1$. Since $E_i\cap E_{i+1}\neq E_{i-1}\cap E_i$ we get that for every $i=1, \ldots, l'$ either $E_i\cap\tilde{E}_i\neq E_{i-1}\cap E_i$ or $E_i\cap\tilde{E}_i\neq E_{i+1}\cap E_i$. Obviously, if e.g. $E_{i-1}\cap E_i\neq E_i\cap\tilde{E}_i$ then $\tilde{E}_{i-1}=\tilde{E}_i$. Consequently, for every $i=1, \ldots, l'$ either $\tilde{E}_{i-1}=\tilde{E}_i$ or $\tilde{E}_i=\tilde{E}_{i+1}$. Thus, we have that the set $\{\tilde{E}_1, \ldots, \tilde{E}_{l'}\}$ contains a cycle of length $<l$ in $(\mathcal{B}_k)_e$, which is a contradiction. ▮

This completes the proof of Proposition 2.2. ▮ ▮

Proof of Theorem 1.3. Repeat the construction given in the proof of Proposition 2.3 with the following inputs: If $l=1$ then put $a=k$ and

$$H' = (\{1, \ldots, \{a\}, \{i, j\}: 1 \leqq i < j \leqq a\}) = K(a).$$

If $l>1$ then let H' be a Ramsey graph for G, $\mathcal{G}'=\binom{H'}{G}$ (H' exists by the induction hypothesis used for $l=1$). If $H=P^m$, $\mathcal{G}=\mathcal{G}^m$ is the result of this construction then the same proof as above establishes Theorem 1.3. ▮

Proof of Theorem 1.4. Assume without loss of generality $n>2$. Put $G=K(n)$ and use for G exactly the same construction as in the proof of Theorem 1.3. Let $H=P^m$ be the resulting graph. It suffices to prove that the family $\binom{H}{K(n)}_e$ does not contain cycles of length $<l$. This can be done by induction on l (the case $l=1$ being trivial). In the inductive step, let $P^0, P^1, \ldots, P^m=H$ be the graphs constructed in the proof of Theorem 1.3. In this situation, we prove by induction on k that $\binom{P^k}{K(n)}_e$ does not contain cycles of length $<l+1$.

This is obvious for $k=0$.

Suppose that some copies $K_1, \ldots, K_{l'}$ of $K(n)$ form a cycle in $\binom{P^{k+1}}{K(n)}_e$. Each K_i is of the form $K_i=B_i*\bar{K}_i$. Let $E_i, \tilde{E}_i, \bar{E}_i$ denote the edge sets of $K_i, B_i, \bar{K}_i$. Then the following are true:

(1) $B_i \neq B_j \Rightarrow E_i \cap E_j = \tilde{E}_i \cap \tilde{E}_j$;

(2) If $B_1=\ldots=B_{l'}=B$ then $\bar{E}_1, \ldots, \bar{E}_{l'}$ form a cycle in P^k (a contradiction).

(3) If not all of $B_1, \ldots, B_{l'}$ coincide then there exists i such that $B_i=B_{i+1}$. Consequently, the set $\{\tilde{E}_1, \ldots, \tilde{E}_{l'}\}$ contains a cycle of length $<l$ in $(\mathscr{B}^{k+1})_e$ (a contradiction).

This completes the proof of Theorem 1.4. ∎

4. Concluding remarks

We can extend Theorem 1.4 to several noncomplete graphs as well. E.g., the following is true.

Definition 4.1. *A graph $G=(V,E)$ is called 3-chromatically connected if the graph $(V-V', \{e\in E\colon e\cap V'=\emptyset\})$ is connected for every bipartite subgraph $(V', \{e\in E\colon e\subseteq V'\})$ of G.*

Theorem 4.2. *Let k, l be positive integers and let G be a 3-chromatically connected graph. Then there exists a graph H with the following properties:*

(i) $H\to(G)_k$

(ii) $\binom{H}{G}_e$ *does not contain cycles of lenght $<l$.*

Conjecture 4.3. *For every graph G and positive integers k, l there exists a graph H such that*

(i) $H\to(G)_k$

(ii) $\binom{H}{G}_e$ *does not contain cycles of length $<l$.*

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